Problem 33

In each of Problems 30 through 35, use the result of Problem 29 to find the Laplace transform of the given function; a and b are real numbers and n is a positive integer.

$$f(t) = t^n e^{at}$$

Solution

The Laplace transform of a function $f(t)$ is defined here as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Substitute the given function and evaluate the integral.

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} t^n e^{at}\, dt \\
&= \int_0^\infty \left[(-1)^n \frac{\partial^n}{\partial s^n} e^{-st}\right] e^{at}\, dt \\
&= (-1)^n \frac{d^n}{ds^n} \int_0^\infty e^{-st} e^{at}\, dt \\
&= (-1)^n \frac{d^n}{ds^n} \int_0^\infty e^{(a-s)t}\, dt \\
&= (-1)^n \frac{d^n}{ds^n} \left[\frac{1}{a-s} e^{(a-s)t}\bigg|_0^\infty\right] \\
&= (-1)^n \frac{d^n}{ds^n} \left(\frac{1}{s-a}\right) \\
&= (-1)^n \frac{d^n}{ds^n} [(s-a)^{-1}] \\
&= (-1)^n [(-1)^n n! (s-a)^{-1-n}] \\
&= (-1)^{2n} \frac{n!}{(s-a)^{n+1}} \\
&= \frac{n!}{(s-a)^{n+1}}
\end{aligned}$$